Complex Integration and Cauchy's Theorem

Sem- $\overline{\text{VI}}$

Paper- C13

Course- Mathematics (H) UG.

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Complex Integration:

The concept of indefinite integral as the process of inverse of differentiation in case of function of real variable. Also it extends to a function of complex variable if the complex function $f(z)$ is analytic.

If $f(z)$ is an analytic function of z and if $\int f(z)dz = F(z)$ then $F'(z) = f(z)$.

However, the concept of definite integral of a function of a real variable does not extend to the domain of complex variable.

In case of real variable, for the definite integral $\int_a^b f(x)\,dx$, the path of integration is always along the x-axis from $x=a$ to $x=b$. Or may be along y-axis of the integral $\int_c^d f(y)\,dy$.

But in case of complex function $f(z)$, the path of definite integral $\int_a^b f(z)\,dz$ can be along any curve from $z=a$ to $z=b$.

Curves in complex plane:

A continuous curve or simply curve or a path in $\mathbb{C}$ is continuous mapping $\gamma : [a,b] \longrightarrow \mathbb{C}$, $a<b$ and $a,b \in \mathbb{R}$. So, the parametric representation of continuous curve γ is given by

$$\gamma(t) = x(t) + i\,y(t),\ t\in[a,b]$$

where $x(t)$ and $y(t)$ are continuous real valued functions of t in the range $a \le t \le b$.

Then the curve is called continuous curve or continuous path or arc.

Here $\gamma(a)$ and $\gamma(b)$ are called initial and terminal points of the curve respectively.

Closed curve: A curve γ with parametric interval $[a,b]$ such that $\gamma(a) = \gamma(b)$ is called a closed curve.

Jordan arc: A continuous arc without multiple points is called a Jordan arc, i.e if $\gamma(t)$ is one to one ie for t_1, t_2 in $[a,b]$, with $t_1 \neq t_2$ we have $\gamma(t_1) \neq \gamma(t_2)$, then $\gamma(t)$ is called Jordan arc.

For example: $\gamma(t) = e^{it}$, $t\in[0,\pi]$ is a Jordan arc.

Simple Curve: A curve defined on $[a,b]$ is called a simple if it does not intersect itself. i.e if $\gamma(t_1) \neq \gamma(t_2)$ for $t_1 \neq t_2$ where the possible exception $\gamma(a) = \gamma(b)$ is allowed.

Regular Curve: A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be continuously differentiable on $[a,b]$ or a curve of class C^1 on $[a,b]$ or simply a C^1 curve on $[a,b]$ if the function $\gamma(t) = x(t) + iy(t)$ continuously differentiable on $[a,b]$

i.e. $x'(t)$ and $y'(t)$ exist on $[a,b]$ and are continuous on $[a,b]$.

Then the curve is called a regular curve or smooth curve.

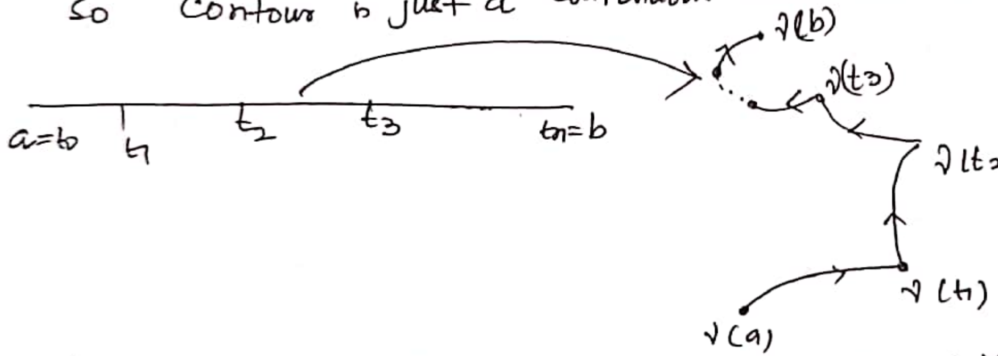
Jordan Curve: A continuous Jordan curve consists of a chain of finite number of continuous arcs.

Contour: Jordan curve consisting of continuous chain of a finite number of regular arcs.

If P be the starting point of the 1st curve and Q be the end point of the last arc then the integral along such a curve is written as $\int_{PQ} f(z)\,dz$.

A curve $\gamma(t)$, $a \le t \le b$ is called piecewise C' (or piecewise smooth curve) if there is a subdivision $a = t_0 < t_1 < t_2 \cdots t_{j-1} < t_n = b$ of the interval $[a,b]$ such that the restriction of γ to each subinterval $[t_j, t_{j+1}]$, $0 \le j \le n-1$ is a smooth curve.

So contour is just a continuous curve that is piecewise smooth

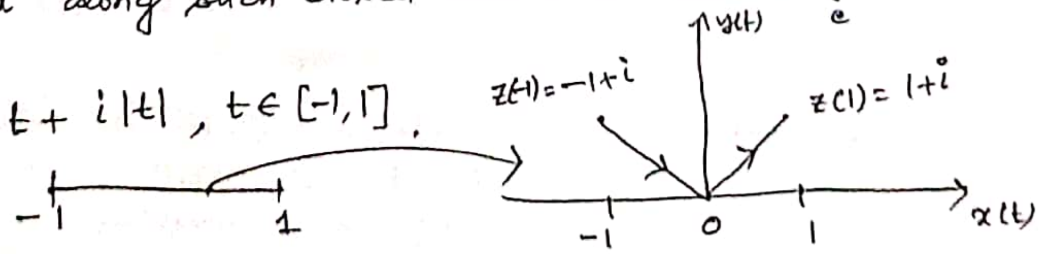


Given a domain D in $\mathbb{C}$, two points z_1 and z_2 in D (need not be distinct) there exist a contour in D with initial point z_1 and terminal point z_2.

The contour is said to be closed if the starting point z_1 of the arc coincides with the terminal point z_2 of the last arc.

The integral along such closed contour is written as $\int_c f(z)\,dz$

Let $\gamma(t) = t + i|t|$, $t \in [-1,1]$.



$$\gamma(t) = t - it \text{ if } t \in [-1,0]$$
$$= t + it \text{ if } t \in [0,1].$$

It is seen that $\gamma(t)$ is smooth over $[-1,0]$ and $[0,1]$ but γ is not smooth because $\gamma'(t)$ fails to exist at $t=0$. $\gamma'(t)$ is discontinuous at 0 but γ is piecewise continuously differentiable since $\gamma'(t) = 1-i$ on $[-1,0)$ and $\gamma'(t) = 1+i$ on $[0,1]$, Accordingly γ is a piecewise smooth curve.

Properties of Complex line integrals:

For a continuous function $F : [a,b] \to \mathbb{C}$ where $F = U + iV$.

We have $\displaystyle\int_a^b F(t)\,dt = \int_a^b U(t)\,dt + i\int_a^b V(t)\,dt$

A complex valued function f is said to be continuous on a continuously differentiable curve $\gamma : [a,b] \to \mathbb{C}$ (or on a contour) if $\varphi(t) = f(z) = f(\gamma(t)) = u(t) + i\,v(t)$ is continuous for $a \le t \le b$.

Suppose f is a complex valued function that is continuous on an open set $D \subseteq \mathbb{C}$ and that $\gamma : [a,b] \to \mathbb{C}$ is a contour with $\gamma([a,b]) \subset D$.

So, the complex integral or contour integral of f along the contour γ, denoted as $\int_\gamma f(z)\,dz$ as

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t))\,\gamma'(t)\,dt$$

Where $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ and $[t_j, t_{j+1}]$, $j = 0,1,2,\ldots,n-1$. being the intervals in which γ is differentiable and the integrals in the sum are Riemann integrals. The contour γ is called the path of the integration of the contour integral.

For example: if $\gamma(t) = z_0 + re^{it}$ is a circle then for an arbitrary continuous function f defined on $\gamma : |z - z_0| = r$,

$$\int_{|z-z_0|=r} f(z)\,dz = \int_0^{2\pi} f(z_0 + re^{it})\, i r e^{it}\,dt$$

$\boxed{\text{Ex}}$: Evaluate $I_j = \int_{\gamma_j} x\,dz$, $j = 1$ to 4 where

(i) γ_1 is a straight line segment from 0 to $a+ib$ $(a, b \in \mathbb{R})$

(ii) γ_2 is the circle $|z| = R$

(iii) γ_3 is the boundary of the square $[0,1] \times [0,1]$ with $\mathbb{C}$ considered as $\mathbb{R}^2$.

(iv) γ_4 is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans: (i) γ_1 may be parameterized by $\gamma_1(t) = (a+ib)t, \ 0 \le t \le 1$.

$$\therefore I_1 = \int_{\gamma_1} [\operatorname{Re} \gamma_1(t)] \gamma_1'(t)\, dt$$

$$= \int_0^1 at(a+ib)\, dt = \frac{a(a+ib)}{2}$$

(ii) Parameterized γ_2 by $\gamma_2(t) = Re^{it}, \ 0 \le t \le 2\pi$.

$$\therefore I_2 = \int_0^{2\pi} (R\cos t)\, iRe^{it}\, dt$$

$$= iR^2 \int_0^{2\pi} [\cos^2 t + i \sin t \cos t]\, dt$$

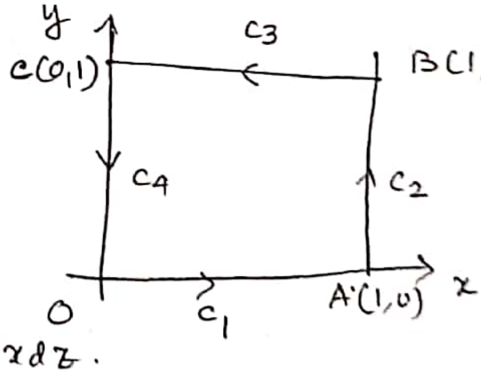
$$= iR^2 \left[\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt + \frac{i}{2} \int_0^{2\pi} \sin 2t\, dt \right]$$

$$= iR^2 \left[\frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{2\pi} + \frac{i}{2}\left(-\frac{\cos 2t}{2}\right)\Big|_0^{2\pi} \right]$$

$$= iR^2 \pi.$$

(iii)

Here the curve γ_3 can be written as



$$\gamma_3 = c_1 + c_2 + c_3 + c_4$$

$$\therefore I_3 = \left(\int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} \right) x\, dz.$$

For the sake of convenience, we parameterize OA, AB, BC and OC as follows,

$c_1(t) = (1-t)\cdot 0 + t\cdot 1 = t$

$c_2(t) = (1-t)\cdot 1 + (1+i)\cdot 1 = 1+it$

$c_3(t) = (1-t)(1+i) + t\cdot i = 1-t+i$

$c_4(t) = (1-t)\, i + t\cdot 0 = (1-t)\, i$ where $t \in [0,1]$.

and $\gamma_3 = c_1 + c_2 + c_3 + c_4$. So we have

$$I_3 = \int_0^1 t\, dt + \int_0^1 1\cdot i\, dt + \int_0^1 (1-t)(-1)\, dt + \int_0^1 0\cdot(-i)\, dt$$

$= \frac{1}{2} + i - \frac{1}{2} = i$

(iv) γ_4 can be written as $\gamma_4(t) = a\cos t + i b\sin t$, $0 \le t \le 2\pi$.

$$\therefore I_4 = \int_0^{2\pi} (a\cos t)(-a\sin t + ib\cos t)\,dt$$

$$= -\frac{a^2}{2}\int_0^{2\pi} \sin 2t \cdot dt + iab\int_0^{2\pi} \cos^2 t\,dt$$

$$= \pi ab\, i$$

Rectifiable Curve :

Let the equation of the arc of a plane curve be $x = \phi(t)$, $y = \psi(t)$ where $a \le t \le b$. Subdivide the interval $[a,b]$ by points $a = t_0, t_1, t_2, \ldots, t_n = b$ and let $P_0, P_1, P_2, \ldots, P_n$ be the points on the curve corresponding to these values of t. The length of the polygonal line $P_0P_1 \ldots P_2P_3 \ldots P_{n-1}P_n$ is the sum of the lengths of the lines

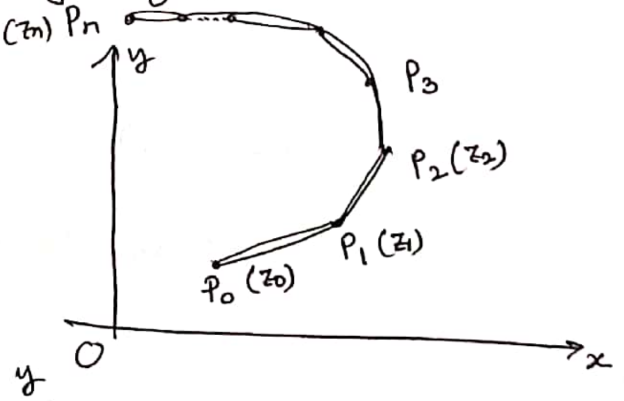
$P_0P_1, P_1P_2, \ldots P_{n-1}P_n$.

Let $z_0, z_1, z_2, \ldots z_n$ be the points on the arc which correspond to the values $t_0, t_1, t_2, \ldots t_n$, then the length of the polygonal arc

$$P_0P_1 \ldots P_n = \sum_{r=1}^{n} \left[(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2\right]^{1/2}$$

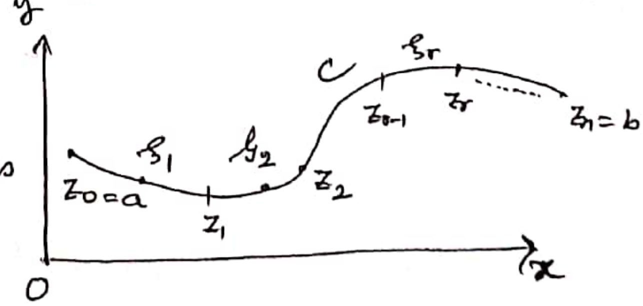
The value of this sum depends upon the mode of subdivisions and is called the length of an inscribed polygon.

If the arc is such that this sum have finite upper bound I, for all modes of sub-divisions, the curve is said to be rectifiable and I is the length of the curve.



Riemann's defn of Integration :

Let a function $f(z)$ of a complex variable z be continuous in a domain D and a, b be two points in this domain D, then



integral of $f(z)$ from a to b defined as below.

Let C be any curve joining a and b and lying entirely in the domain D so that $f(z)$ is continuous on C.

Let there be any partition $P = (a = z_0, z_1, z_2, \dots, z_{r-1}, z_r, \dots z_n = b)$ of the curve C.

From the sum $\sum_{r=1}^{n} (z_r - z_{r-1}) f(\xi_r)$, $z_{r-1} \le \xi_r \le z_r$, the limit of the sum if it exists uniquely, for any path whatsoever joining a, b lying in D and for any mode of division is called the integral of $f(z)$ over C from a to b and is written as

$$\int_C f(z)\,dz = \int_a^b f(z)\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1}) f(\xi_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)\,\Delta z_r \text{ where}$$

$\Delta z_r = z_r - z_{r-1}$ and

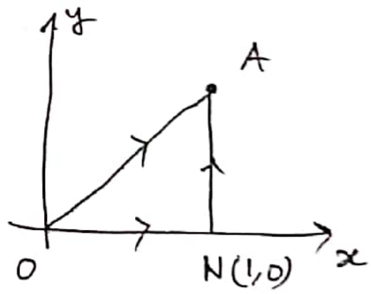
$z_{r-1} \le \xi_r \le z_r$

$\boxed{\text{Ex}}$: Find the value of the integral $\int_0^{1+i} (x - y + ix^2)\,dz$

(a) along the straight line from $z=0$ to $z=1+i$

(b) along the real axis from $z=0$ to $z=1$ and then along the line parallel to the imaginary axis from $z=1$ to $z=1+i$

Ans: (a) Let A be the point of affix $1+i$ and N be the point of affix 1.



(a) Let OA be the line from $z=0$ to $z=1+i$

On OA, $y = x$, $z = x + ix$.

$dz = (1+i)\,dx$.

Hence, $\int_{OA} (x - x + ix^2)(1+i)\,dx$

$$= \int_0^1 ix^2(1+i)\,dx = (-1+i)\left[\frac{x^3}{3}\right]_0^1 = \frac{-1+i}{3}$$

(b) The real axis from $z=0$ to $z=1$ is the line ON and then from $z=1$ to $z=1+i$, a line parallel to imaginary axis, is the line NA.

So, the contour of integration consists of the lines ON and NA.

On ON, $y=0$, $z = x+iy = x$, $dz = dx$.

Hence $\int_{ON} (x - y + ix^2)\,dx = \int_0^1 (x + ix^2)\,dx = \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1$

On the line NA, $x=1$, $z=x+iy=1+iy$, $dz=i\,dy$.

Hence, $\int_{AN}(x-y+ix^2)\,dz$

$$= \int_0^1 (1+i-y)\,i\,dy$$

$$= \left[(-1+i)y - \frac{iy^2}{2}\right]_0^1 = -1+i-\frac{i}{2}$$

$$= -1+\frac{i}{2}$$

Hence $\int_0^{1+i}(x-y+ix^2)\,dz$ along the contour ONA

$$= \int_{OA} + \int_{NA} = \frac{1}{2}+\frac{i}{3}-1+\frac{i}{2}$$

$$= ~~\frac{-1+i}{3}~~ = -\frac{1}{2}+\frac{5}{6}i$$

$\boxed{\text{Ex}}$ Evaluate the integral $\int_0^{1+i} z^2\,dz$

Ans: Here $f(z)=z^2$ is analytic for all finite values of z. Therefore, its integral along a curve joining two fixed points will be the same whatever be the path.

Here we have to integrate z^2 between two fixed points $(0,0)$ and $(1,1)$. Choose the path of integration joining these points as made up of

(i) part of real axis from $(0,0)$ to $(1,0)$. On this line $y=0$, $z=x$, $dz=dx$ and x varies from 0 to 1

(ii) and the line parallel to imaginary axis from the point $(1,0)$ to $(1,1)$. On this line, $x=1$, $z=1+iy$, $dz=i\,dy$ and y varies from 0 to 1.

Thus, $\int_0^{1+i} z^2\,dz = \int_0^1 x^2\,dx + \int_0^1 (1+iy)^2\cdot i\,dy$ along the choosen path.

$$= \frac{1}{3} + \left[\frac{(1+iy)^3}{3}\right]_0^1 = \frac{1}{3}+\frac{(1+i)^3}{3}-\frac{1}{3}$$

$$= \frac{(1+i)^3}{3}$$

$\boxed{\text{Ex}}$: Evalue $\int_{-2+i}^{5+3i} z^3\,dz$.

Here $f(z)=z^3$ is analytic for all finite values of z. So its integration along a curve joining two fixed points will be the same whatever be the path.

Let the path of integration joining these points be along the curve made up of

(i) a line parallel to real axis from the point $(-2,1)$ to the point $(5,1)$. On this line, $y=1$, $z=x+i$, $dz=dx$ and x varies from -2 to 5

(ii) and a line parallel to the axis of imaginary from $(5,1)$ to $(5,3)$. On this line, $x=5$, $z=5+iy$, $dz=i\,dy$. and y varies from 1 to 3.

Thus, $\int_{-2+i}^{5+3i} z^3 dz = \int_{-2}^{5} (x+i)^3 dx + \int_{1}^{3} (5+iy)^3 i\,dy$

$$= \left[\frac{(x+i)^4}{4}\right]_{-2}^{5} + \left[\frac{(5+iy)^4}{4}\right]_{1}^{3}$$

$$= \frac{1}{4}\left[(5+i)^4 - (i-2)^4\right] + \frac{1}{4}\left[(5+3i)^4 - (5+i)^4\right]$$

$\boxed{\text{Ex}}$: Evaluate $\int_c (z^2+3z+2)\,dz$ where c is the arc of the cycloid $x=a(\theta+\sin\theta)$ and $y=a(1-\cos\theta)$ between two points $(0,0)$ and $(\pi a, 2a)$.

Ans: Here the function $f(z)=z^2+3z+2$ is a polynomial and so $f(z)$ is analytic for all value of z. Hence, the integral between two points $(0,0)$ and $(\pi a, 2a)$ is independent of the path joining these points.

For ~~our~~ convenience, the path of integration consisting of

(i) the part of real axis from the point $(0,0)$ to the point $(\pi a, 0)$. On this line, $y=0$, $z=x$, $dz=dx$. and x varies from 0 to πa.

(ii) and the line parallel to imaginary axis from the point $(\pi a, 0)$ to $(\pi a, 2a)$. On this line, $x=\pi a$, $z=\pi a+iy$, $dz=i\,dy$. and y varies from 0 to $2a$.

Hence, $\int_c (z^2+3z+2)\,dz$

$$= \int_0^{\pi a} (x^2+3x+2)\,dx + \int_0^{2a} \left\{(\pi a+iy)^2 + 3(\pi a+iy) + 2\right\} i\,dy$$

$$= \left[\frac{(\pi a)^3}{3} + \frac{3(\pi a)^2}{2} + 2\cdot\pi a\right] + \left[\frac{(\pi a+iy)^3}{3} + \frac{3(\pi a+iy)^2}{2} + 2iy\right]_0^{2a}$$

$$= \frac{(\pi a)^3}{3} + \frac{3}{2}(\pi a)^2 + 2\pi a + \frac{1}{3}(\pi a+2ai)^3 + \frac{3}{2}(\pi a+2ai)^2 + 4i - \frac{(\pi a)^3}{3} - \frac{3(\pi a)^2}{2}$$

$= 2\pi a + \frac{1}{3}(\pi a + 2ai)^3 + \frac{3}{2}(\pi a + 2ai)^2 + 4i$

$\boxed{Ex}$: Find the value of the integral $\int \frac{1}{z-a}\,dz$ round the circle whose equation is $|z-a| = \rho$.

Ans : On the circle $|z-a| = \rho$,

$z - a = \rho e^{i\theta}$, $dz = \rho i e^{i\theta}$, θ varies from 0 to 2π

Hence, $\int_c \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{\rho e^{i\theta}}{\rho e^{i\theta}}\, i\,d\theta = 2\pi i$

$\boxed{Ex}$: Evaluate $\int_c \bar{z}\,dz$ from $z=0$ to $z = 4+2i$ along the curve c given by (a) $z = t^2 + it$ (b) the line from $z=0$ to $z=2i$ and then the line from $z = 2i$ to $z = 4+2i$

Ans : (a) The point $z=0$ to $z = 4+2i$ on c corresponds to $t=0$ and $t=2$ respectively. Then the line integral becomes

$$\int_{t=0}^{2} \overline{t^2 + it}\; d(t^2 + it)$$

$$= \int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 - it^2 + t)\,dt = 10 - \frac{8i}{3}.$$

Another way: The given integral equals to

$$\int_c (x - iy)(dx + idy) = \int_c x\,dx + y\,dy + i\int_c x\,dy - y\,dx$$

The parametric equation of c is given by $x = t^2$, $y = t$. So, t ranges from $t=0$ to $t=2$. Then the line integral equals

$$\int_{t=0}^{2} (t^2\, 2t\,dt + t\,dt) + i\int_0^2 (t^2\,dt - t \cdot 2t\,dt)$$

$$= \int_0^2 (2t^3 + t)\,dt + i\int_0^2 (-t^2)\,dt$$

$$= 10 - \frac{8i}{3}$$

(b) The given integral equals

$$\int_c (x - iy)(dx + idy) = \int_c x\,dx + y\,dy + i\int_c x\,dy - y\,dx$$

The line from $z=0$ to $z = 2i$ is the same as the line from $(0,0)$ to $(0,2)$ for which $x=0$, $dx=0$ and the line integral equals

$$\int_{y=0}^{2} 0+y\,dy + i\int_{y=0}^{2} 0(dy)-y\cdot 0 = \int_0^2 y\,dy = 2.$$

The line from $z=2i$ to $z=4+2i$ is the same as the line from $(0,2)$ to $(4,2)$ for which $y=2$ and $dy=0$, and the line integral equals

$$\int_{x=0}^{4} x\,dx + i\int_{x=0}^{4}(-2dx)$$

$$= \int_0^4 x\,dx - 2i\int_0^4 dx = 8-8i$$

Then the required value is $2+8-8i$
$= 10-8i$

Theorem: If a function $f(z)$ is continuous on a contour c of length L and if M be the upper bound of $|f(z)|$ on c then

$$\left|\int_c f(z)dz\right| \le ML.$$

Proof: Let the equation of the curve c be $x=\varphi(t)$, $y=\psi(t)$.

$$\therefore L = \int ds = \int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \quad \text{—(1)}$$

$$z = x+iy, \quad dz = dx+idy$$

so $\quad |dz| = \sqrt{(dx)^2+(dy)^2}$

$$\therefore \int|dz| = \int\sqrt{(dx)^2+(dy)^2}.$$

$$= \int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = L \quad \left[\text{from (1)}\right] \quad \text{(2)}$$

We have $\displaystyle\int_c f(z)dz = \lim_{n\to\infty}\sum_{r=1}^{n} f(\xi_r)(z_r-z_{r-1}).$

Now, $\displaystyle\left|\sum_{r=1}^{n} f(\xi_r)(z_r-z_{r-1})\right| \le \sum_{r=1}^{n}|f(\xi_r)|\,|z_r-z_{r-1}|.$

By making $n\to\infty$ the above inequality may be written as

$$\left|\int_c f(z)dz\right| \le \int_c |f(z)|\,|dz|$$

$$\le M\int_c |dz| \quad \left[\because \text{Max}|f(z)| = M\right]$$

$= ML$ [using (2)]

Thus, $\left| \int_c f(z) dz \right| \le ML$. [This is an upper bound for a complex integral]

Green theorem in a plane:

If be the closed region of the xy plane bounded by a simple closed curve c and if M and N be functions of x and y which are continuous, having continuous derivatives in S then

$$\oint_c M dx + N dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$ [Proof is done in previous multiple integral]

where c is traversed in a counter clock wise sense i.e. in the direction of travel round c in which the interior of S lies on the left.

Ex: Let $P(x,y)$ and $Q(x,y)$ be continuous and have continuous first partial derivatives at each point of a simply connected region R. Prove that a necessary and sufficient condition that $\oint_c P dx + Q dy = 0$ around every closed path c in R is that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ identially in R.

Ans:

Sufficient condition: Suppose $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then by Green's theorem

$$\oint_c P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = 0,$$

where R is the region bounded by c.

Necessary Condition: Suppose $\oint_c P dx + Q dy = 0$ around every closed path c in R and that

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ at some point in R.

ie $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} > 0$ at the point (x_0, y_0).

By hypothesis, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous in R so that there must some region Ω containing (x_0, y_0) as an interior point for which $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} > 0$. If Γ is the boundary of Ω then by Green's Theorem $\oint_{\Gamma} Pdx + Qdy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy > 0.$

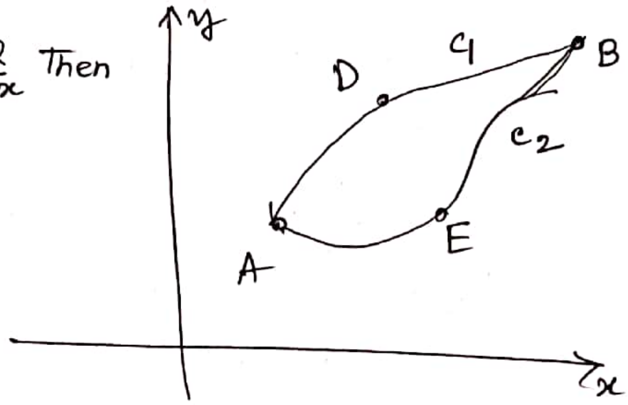
Contradicting the hypothesis that $\oint_C Pdx + Qdy = 0$ for all closed curves in R. So $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ can not be positive.

Similarly, we can show that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ cannot be negative and it follows that it must be identically zero.

i.e $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ identially in R.

Hence the result.

$\boxed{\text{Ex}}$: Let $P(x,y)$ and $Q(x,y)$ be continuous and have continuous first partial derivatives at each point of a simply conneted region R. Prove that a necessary and sufficient condition that $\int_A^B Pdx + Qdy$ be independent of the path in $\mathbb{R}$ joing the points A and B is that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ identically in R.



Sufficient condition: If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ Then

$$\int_{ADBEA} Pdx + Qdy = 0.$$

$$\Rightarrow \int_{ADB} + \int_{BEA} = 0$$

$$\Rightarrow \int_{ADB} = -\int_{BEA} = \int_{AEB}$$

$$\Rightarrow \int_{ADB} = \int_{AEB} \Rightarrow \int_{c_1} = \int_{c_2}$$

i.e. the integral is independent of the path.

Necessary condition: If the integral is independent of the path, then for all paths c_1 and c_2 in R we have

$$\int_{c_1} = \int_{c_2} \Rightarrow \int_{ADB} = \int_{AEB} \Rightarrow \int_{ADBEA} = 0$$

From this it follows that the integral around any closed path in R is zero, and by previous example, that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Hence the result.

$\boxed{\text{Ex}}$: Prove that equivalence of the operators (i) $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$

Where $z = x+iy$, $\bar{z} = x-iy$ (ii) $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$

Ans: Let F be a continuously differentiable function of x and y.

$$\therefore \frac{\partial F}{\partial x} = \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} + \frac{\partial F}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial x}$$

$$= \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \bar{z}} \qquad \therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}.$$

Also,
$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} + \frac{\partial F}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial y}$$

$$= i\frac{\partial F}{\partial z} - i\frac{\partial F}{\partial \bar{z}}$$

$$= i\left(\frac{\partial F}{\partial z} - \frac{\partial F}{\partial \bar{z}}\right) \qquad \therefore \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right).$$

$\boxed{\text{Ex}}$: Show that (i) $\vec{\nabla} \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{z}}$

Ans: Now,
$$\vec{\nabla} \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

$$\equiv \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} + i\left(i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}\right)$$

$$\equiv 2\frac{\partial}{\partial \bar{z}}.$$

$\boxed{\text{Ex}}$: Show that $\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) = 2\frac{\partial B}{\partial \bar{z}}$

where $B(z, \bar{z}) = P(x,y) + iQ(x,y)$.

Ans: We have $\vec{\nabla} \equiv 2\frac{\partial}{\partial \bar{z}}$

$$\therefore \vec{\nabla} B = 2\frac{\partial B}{\partial \bar{z}}$$

Hence
$$\vec{\nabla} B = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ) = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)$$

$$= 2\frac{\partial B}{\partial \bar{z}}$$

Green's Theorem in Complex form:

If $B(z, \bar{z})$ is continuous and has continuous partial derivatives in a region R and on its boundary C where $z = x+iy$, $\bar{z} = x-iy$, Prove that Green's theorem can be written in complex form as

$$\oint_C B(z,\bar{z})\, dz = 2i \iint_R \frac{\partial B}{\partial \bar{z}}\, dx\,dy.$$

Proof: Let $B(z,\bar{z}) = P(x,y) + iQ(x,y)$. Then using Green's theorem, we have

$$\oint_C B(z,\bar{z})\, dz = \oint_C (P+iQ)(dx+idy)$$

$$= \oint_C P\,dx - Q\,dy + i\oint_C Q\,dx + P\,dy$$

$$= -\iint_R \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$$

$$= i\iint_R \left[\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right] dx\,dy$$

$$= 2i\iint_R \frac{\partial B}{\partial \bar{z}}\, dx\,dy.$$

Hence Proof the Theorem.

Cauchy's Theorem:

If $f(z)$ is analytic function of z and $f'(z)$ is continuous at all points inside and on a simple closed curve C then

$$\int_C f(z)\, dz = 0.$$

Proof: Since $f(z) = u+iv$ is analytic and has continuous derivatives at all points inside and on a simple closed curve C.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad [\because u_x = v_y \text{ and } u_y = -v_x]$$

Since u, v, u_x, v_x, u_y, v_y are all continuous inside and on C.

Thus, Green's theorem can be applied and we have

$$\oint_C f(z)\, dz = \oint_C (u+iv)(dx+idy) = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

[By C-R equations $u_x = v_y$ and $u_y = -v_x$]

Hence, $\oint_C f(z)\,dz = 0$

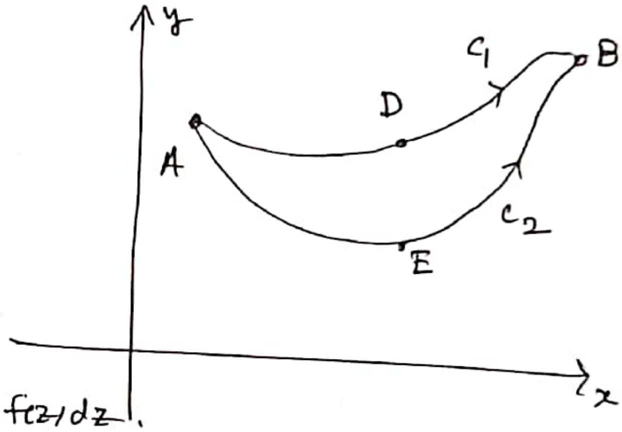
$\boxed{\text{Ex}}$: Prove that (i) $\oint_C dz = 0$ (ii) $\oint_C z\,dz = 0$ (iii) $\oint_C (z - z_0)\,dz = 0$ Where C is any simple closed curve and z_0 is a constant.

Ans : (i) Here the function $f(z) = 1$ is analytic inside C and has a continuous derivative. Hence by Cauchy's theorem,

$$\oint_C dz = 0.$$

(ii), (iii) Try yourself.

$\boxed{\text{Ex}}$: If $f(z)$ is analytic in a simply connected region R, Prove that $\int_a^b f(z)\,dz$ is independent of the path in R joining any two points a and b in R.



Ans : We have by Cauchy's theorem,

$$\int_{ADBEA} f(z)\,dz = 0$$

$$\Rightarrow \int_{ADB} f(z)\,dz + \int_{BEA} f(z)\,dz = 0$$

Hence, $\int_{ADB} f(z)\,dz = -\int_{BEA} f(z)\,dz = \int_{AEB} f(z)\,dz.$

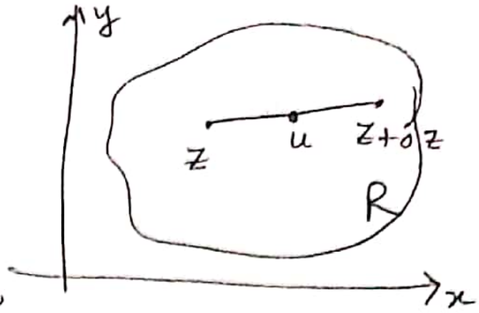
Thus, $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = \int_a^b f(z)\,dz.$

This shows that $\int_a^b f(z)\,dz$ is independent of the path in R joining any two points a and b in R.

$\boxed{\text{Ex}}$: Let $f(z)$ is analytic in a simply connected region R and let a and z be points in R. Prove that (i) $F(z) = \int_a^z f(u)\,du$ is analytic in R and (ii) $F'(z) = f(z)$.

Ans : We have $\dfrac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \dfrac{1}{\Delta z}\left[\int_a^{z+\Delta z} f(u)\,du - \int_a^z f(u)\,du\right] - f(z)$

$$= \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(u) - f(z)]\, du \quad \text{—} (1)$$



By Cauchy's Theorem, the integral is independent of the path joining z and $z+\Delta z$ so long as the path is in R. In particular, we can choose the path as line segment joining z and $z+\Delta z$, provided we choose $|\Delta z|$ small enough so that this path lies in R.

Now, by continuity of $f(z)$, we have for all points u on this straight line path, $|f(u) - f(z)| < \epsilon$ whenever $|u - z| < \delta$, $|\Delta z| < \delta$ $(\delta > 0)$.

Further more, $$\left| \int_{z}^{z+\Delta z} |f(u) - f(z)|\, |du| \right| < \epsilon |\Delta z| \quad \text{—} (2)$$

So that from (1) $$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_{z}^{z+\Delta z} [f(u) - f(z)]\, du \right|$$

$$< \epsilon$$

for $|\Delta z| < \delta$.

This however, saying that $\lim\limits_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$

ie $F(z)$ is analytic and $F'(z) = f(z)$.

$\boxed{\text{Ex}}$: If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points $(1,1)$ and $(2,3)$, find the value of $\int_C (12z^2 - 4iz)\, dz$.

Ans : Since the integral is independent of the path joining $(1,1)$ and $(2,3)$. Hence any path can be chosen. So let us choose the straight line from $(1,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,3)$.

Along the line from $(1,1)$ to $(2,1)$: In this line, $y = 1$, $dy = 0$
$z = x + i$ and $dz = dx$ and varies from 1 to 2.

Then the integral equals

$$\int_{x=1}^{2} [12(x+i)^2 - 4i(x+i)]\, dx$$

$$= \left[4(x+i)^3 - 2i(x+i)^2\right]_1^2 = 20 + 30i$$

Along the line from $(2,1)$ to $(2,3)$: On this line, $x = 2$, $dx = 0$

$z = 2+iy$, $dz = i\,dy$.

Then the integral becomes

$$\int_1^3 \left[12(2+iy)^2 - 4i(2+iy)\right] i\,dy$$

$$= \left[4(2+iy)^3 - 2i(2+iy)^2\right]_1^3 = -176 + 8i$$

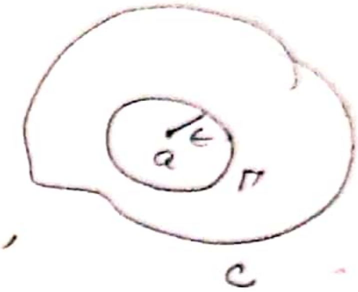
$$\therefore \int_C (12z^2 - 4iz)\,dz = 20 + 30i - 176 + 8i = -156 + 38i$$

Another way: The given integral becomes

$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\,dz = \left[4z^3 - 2iz^2\right]_{1+i}^{2+3i} = -156 + 38i.$$

Ex : Evaluate $\oint_C \frac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots$ where $z = a$ is inside the simple closed curve C.

Ans : Since a is inside C and let Γ be a circle of radius ϵ with centre at $z = a$ so that Γ is inside C.



By Cauchy's Theorem for multiconnected region,

$$\oint_C \frac{dz}{(z-a)^n} = \int_\Gamma \frac{dz}{(z-a)^n} \qquad \left[\Gamma:\ z - a = \epsilon e^{i\theta},\ dz = \epsilon i e^{i\theta}\right]$$

$$= \int_0^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta}{\epsilon^n \cdot e^{in\theta}} = \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta}\,d\theta$$

$$= \frac{i}{\epsilon^{n-1}} \left[\frac{e^{(1-n)i\theta}}{(1-n)i}\right]_0^{2\pi}$$

$$= \frac{1}{(1-n)\epsilon^{n-1}} \left[e^{(1-n)2\pi i} - 1\right] = 0, \quad \text{when } n \neq 1.$$

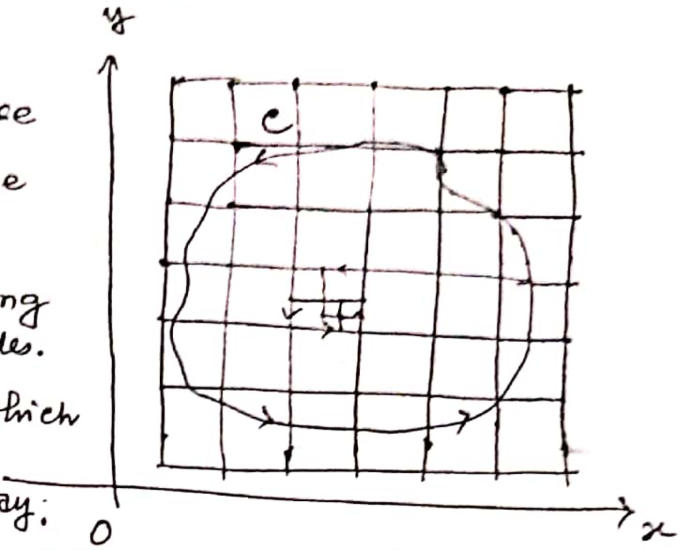
Cauchy Goursat theory:

If a function $f(z)$ is analytic and one valued inside and on a simple closed contour C then $\int_C f(z)\,dz = 0$

Proof: Before going to proof of this theorem, we need ~~to~~ prove the lemma :-

Lemma-1 : Given ϵ, it is possible to divide the region inside

C into finite number of meshes, either complete squares or parts of squares, such that within each mesh, ~~point~~ there exists a point z_0 such that $\left|\dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$ for all values of z in the mesh. — (1)



__Proof:__ Suppose the lemma is false then it fails atleast in one mesh.

Subdivide this mesh by lines joining the middle points of the opposite sides.

If there still remain any parts which do not satisfy the condition (1). We again subdivide in the same way.

The process may end either after a finite number of steps when the condition (1) is satisfied for every subdivision or the process may go on indefinitely.

For the 2nd case, we obtain a sequence of squares whose limit point is z_0 which lies inside C and at which the condition (1) is not satisfied at the point z_0. Therefore,

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \nless \epsilon \quad \text{where } |z-z_0| \text{ is small.}$$

This shows that $f(z)$ is not differentiable at z_0 i.e. $f(z)$ is not analytic at all points within and on the contour C.

This contradicts the above theorem.

So, $$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$$

$$\Rightarrow \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) = \eta(z) \quad \text{where } |\eta| < \epsilon.$$

and $\eta \to 0$ as $z \to z_0$.

Thus, $f(z) = f(z_0) + (z-z_0)f'(z_0) + (z-z_0)\eta(z)$. — (2)

Also we know that $\oint_C dz = 0$ and $\oint_C z\,dz = 0$ — (3)

taken over any continuous closed curve.

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From (2)

Now, $$\oint_C f(z)\,dz = \oint_C z_0\,dz + \oint_C (z-z_0) f'(z_0)\,dz + \oint_C (z-z_0)\eta\,dz$$

$$= 0 + 0 + \oint_C (z-z_0)\eta\,dz \quad [\text{using } (3)] \qquad (4)$$

Let $C_1, C_2, \dots C_m$ be the complete squares and $P_1, P_2, \dots, P_n$ be the partial squares, parts of whose boundaries are parts of C.

Consider $$\sum_{m=1}^{M} \int_{C_m} f(z)\,dz + \sum_{n=1}^{N} \int_{P_n} f(z)\,dz.$$

Where integral along each contour is taken in anti clock-wise direction.

In the complete sum the integration along each straight side of each square (whether complete or partial) happens to be taken twice in opposite directions. Hence, all the integrals along straight boundaries of the squares cancel and only those integrals remain which are taken along the curved boundaries of the partial squares because these are described only once.

The integrals which are left behind sum to

$$\oint_C f(z)\,dz.$$

Thus, $$\oint_C f(z)\,dz = \sum_{m=1}^{M} \oint_{C_m} f(z)\,dz + \sum_{n=1}^{N} \oint_{P_n} f(z)\,dz$$

$$= \sum_{m=1}^{M} \oint_{C_m} (z-z_0)\eta\,dz + \sum_{n=1}^{N} \oint_{P_n} (z-z_0)\eta\,dz \qquad (5)$$

[using (4)]

Where z_0 represents any point inside the boundary C_m or P_n and z represent any point lying on these boundaries.

Now, $$\left|\oint_{C_m} (z-z_0)\eta\,dz\right| \le \oint_{C_m} |z-z_0|\,|\eta|\,|dz|$$

$$\le l\sqrt{2}\,\epsilon \oint_{C_m} |dz| \quad \left[\because |z-z_0| < \sqrt{2}\,l \;\&\; |\eta| < \epsilon\right]$$

$$= l\sqrt{2}\,\epsilon\, 4l \quad [\text{Perimeter of the square}$$

$$= 4\sqrt{2}\,\epsilon\, A_m \qquad \left[\text{Since } l^2 = A_m = \text{area of the square } c_m\right].$$

Similarly, $\left| \oint_{P_n} (z-z_0)\,\eta\, dz \right|$

$$\leq l\sqrt{2}\,\epsilon \int_{P_n} |dz|$$

$$\leq l\sqrt{2}\,\epsilon\,(4l + S_n) \qquad \left[S_n \text{ is the length of the arc } c \text{ which forms curved boundary of } P_n\right]$$

$$= \sqrt{2}\,\epsilon\,(4A_n + l S_n) \qquad \left[A_n = l^2 = \text{area of the square } P_n\right]$$

From (5)

$$\therefore \left| \oint_c f(z)\, dz \right| = \left| \sum_{m=1}^{M} \oint_{c_m} (z-z_0)\,\eta\, dz + \sum_{n=1}^{N} \oint_{P_n} (z-z_0)\,\eta\, dz \right|.$$

$$\leq \sum_{m=1}^{M} \oint_{c_m} \left|(z-z_0)\,\eta\, dz\right| + \sum_{n=1}^{N} \oint_{P_n} \left|(z-z_0)\,\eta\, dz\right|$$

$$\leq 4\sqrt{2}\,\epsilon \sum_{m=1}^{M} A_m + \epsilon\sqrt{2} \sum_{n=1}^{N} (4A_n + l S_n).$$

$$= 4\sqrt{2}\,\epsilon \left[\sum_{m=1}^{M} A_m + \sum_{n=1}^{N} A_n\right] + \epsilon\sqrt{2}\, l \sum_{n=1}^{N} S_n$$

$$= 4\sqrt{2}\,\epsilon A + \epsilon\sqrt{2}\, l L, \qquad \left[\because L = \sum_{n=1}^{N} S_n \text{ (total length of } c\text{)},\ \sum_{m=1}^{M} A_m + \sum_{n=1}^{N} A_n = A\right]$$

$$\longrightarrow 0 \text{ as } \epsilon \to 0.$$

Hence $\oint_c f(z)\, dz = 0$.

<u>Cauchy's theorem for multi-connected region:</u>

<u>Th</u>: If c is closed curve and $c_1, c_2, \ldots$ be the other closed curves which lie inside c and if a function $f(z)$ is analytic in the region between these curves and continuous on c

then $$\int_c f(z)\, dz = \int_{c_1} f(z)\, dz + \int_{c_2} f(z)\, dz + \cdots \quad \cdots$$

where the integral along each curve is taken in the anticlockwise direction.